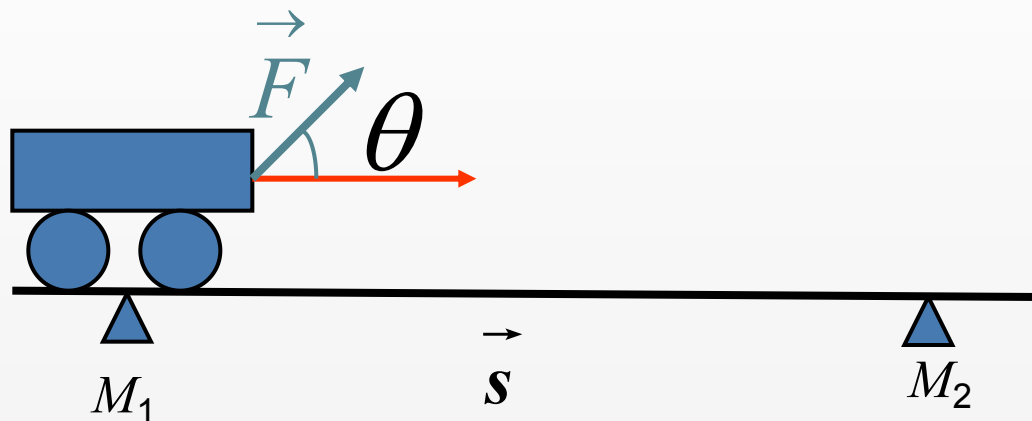


实例

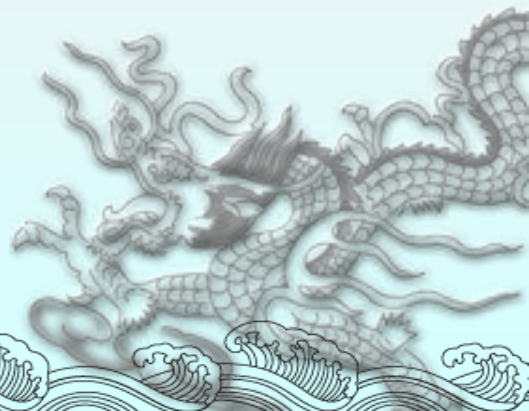


一物体在常力 \vec{F} 作用下沿直线从点 M_1 移动到点 M_2 ，以 \vec{s} 表示位移，则力 \vec{F} 所作的功为

$$W = |\vec{F}| |\vec{s}| \cos \theta \quad (\text{其中} \theta \text{为} \vec{F} \text{与} \vec{s} \text{的夹角})$$

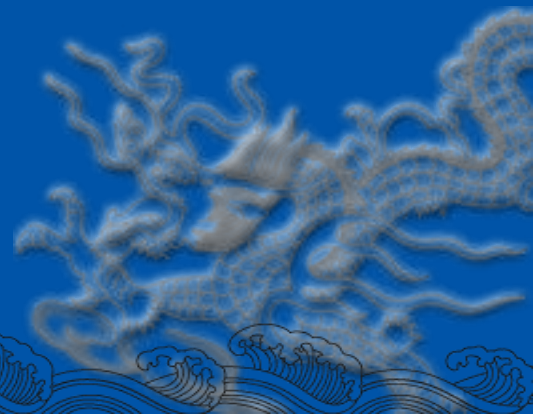
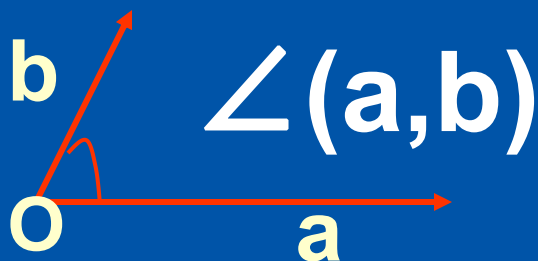
启示 两向量作这样的运算，结果是一个数量。

1.7 两矢量的数量积



定义1.7.1 两个矢量**a**,**b**的模和它们夹角余弦的乘积叫做矢量**a**和**b**的数量积(也称内积、点积), 记为**a**·**b** 或 **ab**, 即

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \angle(\mathbf{a}, \mathbf{b}) \quad (1.7-1)$$



几点说明:

在 $a \cdot b = |a| |b| \cos \angle(a, b)$ 中

- 1) $a \cdot b$ 是一个数量而不是矢量.
- 2) 当 $a=0$ 或 $b=0$ 时, $a \cdot b=0$.
- 3) 当 $\angle(a, b) = \frac{\pi}{2}$ 时, $a \cdot b=0$.

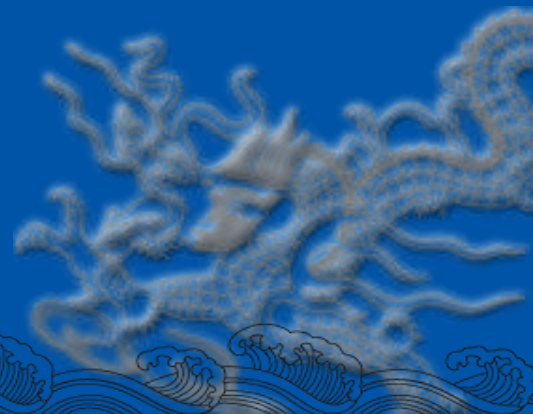
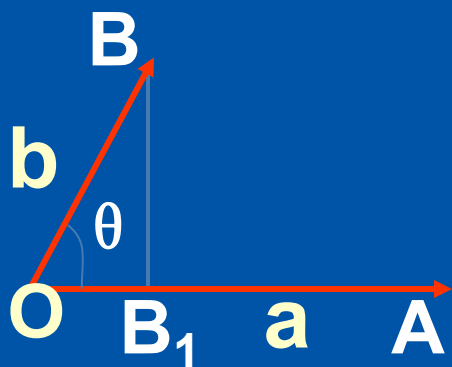
4) 当 \mathbf{a} , \mathbf{b} 是两个非零向量时,

$$|\mathbf{b}| \cos \angle(\mathbf{a}, \mathbf{b}) = \text{射影}_{\mathbf{a}} \mathbf{b}$$

$$|\mathbf{a}| \cos \angle(\mathbf{a}, \mathbf{b}) = \text{射影}_{\mathbf{b}} \mathbf{a}$$

于是从 $ab = |\mathbf{a}| |\mathbf{b}| \cos \angle(\mathbf{a}, \mathbf{b})$ 中, 有

$$ab = |\mathbf{a}| \text{射影}_{\mathbf{a}} \mathbf{b} = |\mathbf{b}| \text{射影}_{\mathbf{b}} \mathbf{a} \quad (1.7-2)$$



5) 当**b**是单位矢量**e**时, $|\mathbf{b}|=|\mathbf{e}|=1$

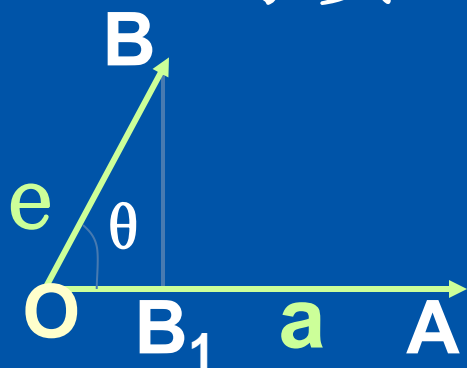
由 $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\angle(\mathbf{a},\mathbf{b})$ 有

$$\mathbf{a}\cdot\mathbf{e}=\text{射影}_e\mathbf{a} \quad (1.7-2')$$

6) 当**a=b**时, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|^2$

事实上, $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}||\mathbf{a}|\cos\angle(\mathbf{a},\mathbf{a})=|\mathbf{a}|^2$

$\mathbf{a}\cdot\mathbf{a}$ 叫做矢量**a**的数量平方.



记作 \mathbf{a}^2 ,即 $\mathbf{a}^2=|\mathbf{a}|^2$

定理1.7.1 两矢量 \mathbf{a}, \mathbf{b} 相互垂直的充要条件是:

$$\mathbf{a} \cdot \mathbf{b} = 0$$

证明:

1) 证必要性 设 $\mathbf{a} \perp \mathbf{b}$, 要证明 $\mathbf{a} \cdot \mathbf{b} = 0$ 成立.

事实上, $\mathbf{a} \perp \mathbf{b}$, 则 $\cos \angle(\mathbf{a}, \mathbf{b}) = 0$, 所以

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \angle(\mathbf{a}, \mathbf{b}) = 0$$

必要性得证.

2) 证充分性 设 $\mathbf{a} \cdot \mathbf{b} = 0$, 要证明 $\mathbf{a} \perp \mathbf{b}$ 成立.

事实上, 因为

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \angle(\mathbf{a}, \mathbf{b}) = 0$$

所以, 有

$$|\mathbf{a}| = 0 \text{ 或 } |\mathbf{b}| = 0 \text{ 或 } \cos \angle(\mathbf{a}, \mathbf{b}) = 0$$

于是 $\mathbf{a} \perp \mathbf{b}$ 成立. 充分性得证.

定理1.7.2 矢量的数性积满足下面的运算规律:

1) 交换律 $a \cdot b = b \cdot a$ (1.7-3)

2) 关于数因子的结合律

$$(\lambda a) \cdot b = \lambda(a \cdot b) = a \cdot (\lambda b) \quad (1.7-4)$$

3) 分配律 $(a+b) \cdot c = a \cdot c + b \cdot c$ (1.7-5)

证: 若(1.7-3), (1.7-4), (1.7-5)中有零矢量,

则它们成立.



若 $a \neq 0$ 且 $b \neq 0$ 且 $c \neq 0$,

1) 证交换律 $a \cdot b = b \cdot a$

由定义, $a \cdot b = |a| |b| \cos \angle(a, b)$

$b \cdot a = |b| |a| \cos \angle(b, a)$

又 $\angle(a, b) = \angle(b, a)$

所以, $a \cdot b = b \cdot a$. 交换律成立.

2) 证关于数因子的结合律

$$(\lambda a).b = \lambda(a.b) = a.(\lambda b)$$

1. $\lambda=0$, 等式显然成立.

2. $\lambda \neq 0$, 则

$$(\lambda a).b = |b| \text{射影}_b(\lambda a) = |b| (\lambda \text{射影}_b a) = \lambda(a.b)$$

$$\text{又 } a.(\lambda b) = (\lambda b).a = \lambda(a.b)$$

所以, $(\lambda a).b = \lambda(a.b) = a.(\lambda b)$ 成立.

3) 分配律

$$ab = |b| \text{射影}_a b = |a| \text{射影}_b a \quad (1.7-2)$$

$$\text{射影}_l(a+b) = \text{射影}_l a + \text{射影}_l b \quad (1.6-3)$$

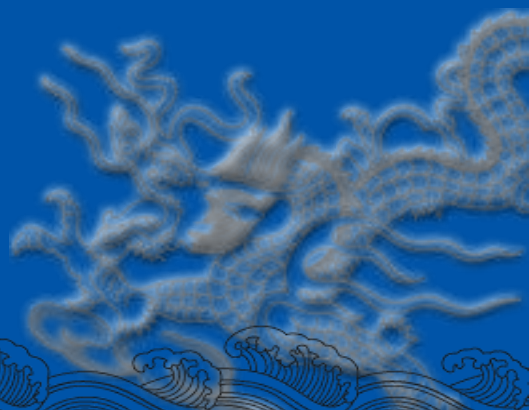
所以 $(a+b)c = |c| \text{射影}_c(a+b)$

$$= |c| (\text{射影}_c a + \text{射影}_c b)$$

$$= |c| \text{射影}_c a + |c| \text{射影}_c b$$

$$= a.c + b.c$$

问题得证.



综上所述,以下公式得证:

1) 交换律 $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (1.7-3)

2) 关于数因子的结合律

$$(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b}) \quad (1.7-4)$$

3) 分配律 $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$ (1.7-5)

推论 $(\lambda \mathbf{a} + \mu \mathbf{b}) \cdot \mathbf{c} = \lambda(\mathbf{a} \cdot \mathbf{c}) + \mu(\mathbf{b} \cdot \mathbf{c})$ (1.7-7)

根据上述运算规律,在矢量数性积的运算中,
可以像多项式的乘法一样去进行.

例如:

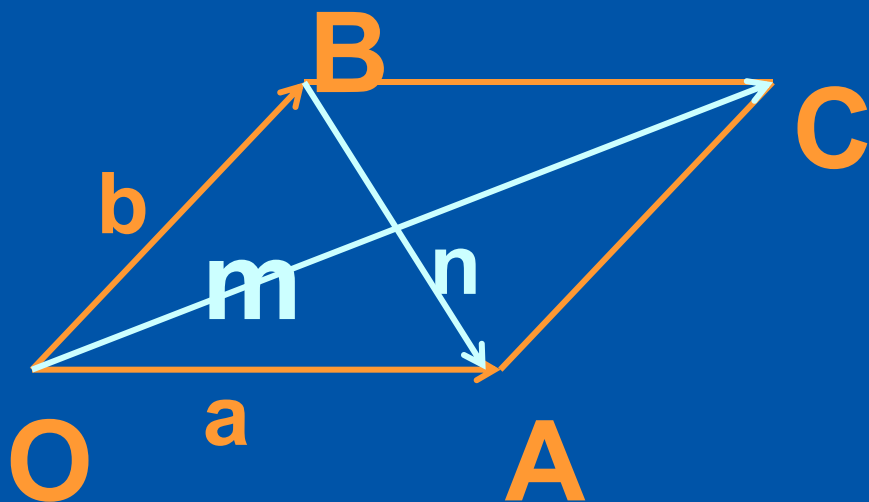
$$(a+b)(a-b)=a^2-ab+ba-b^2= a^2-b^2$$

$$(a \pm b)^2=(a \pm b)(a \pm b)=a^2 \pm ab \pm ba+b^2$$
$$= a^2 \pm 2ab+b^2$$

$$(2a+3b)(c-4d)=2a.c+3b.c-8a.d-12b.d$$

例1 证明平行四边形对角线的平方和等于它各边的平方和.

证: 如图



$$\therefore m = a + b, n = a - b$$

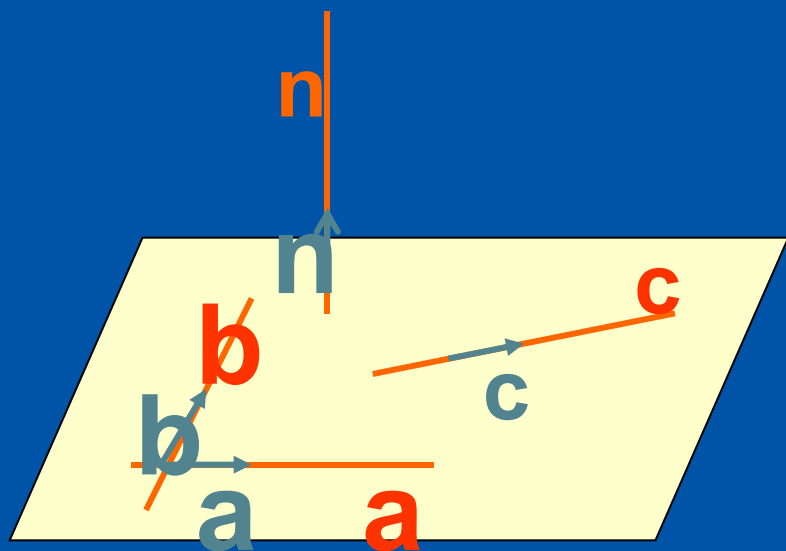
$$\therefore m^2 = (a + b)^2 = a^2 + 2ab + b^2$$

$$n^2 = (a - b)^2 = a^2 - 2ab + b^2$$

$$\therefore m^2 + n^2 = 2(a^2 + b^2) = 2a^2 + 2b^2$$

问题得证.

例2 证明 如果一条直线与一个平面内的两条相交直线都垂直,则它就和平面内任何直线都垂直,即它垂直于平面.



证: 如图在直线 $n, a, b, c,$ 上分别取非零向量 $\mathbf{n}, \mathbf{a}, \mathbf{b}, \mathbf{c},$ 由假设, 有 $\mathbf{n} \perp \mathbf{a}, \mathbf{n} \perp \mathbf{b}$ 问题转化为证明 $\mathbf{n} \perp \mathbf{c}.$

即证 $\mathbf{n} \cdot \mathbf{c} = 0.$

例2 证明 如果一条直线与一个平面内的两条相交直线都垂直,则它就和平面内任何直线都垂直,即它垂直于平面.

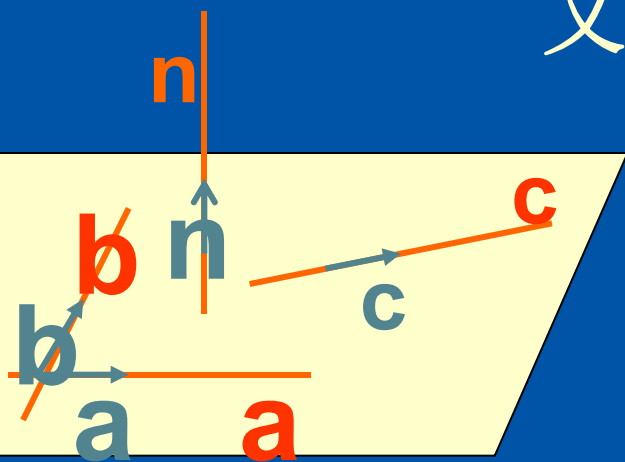
事实上, $\because a, b, c$ 共面, $\therefore c = \lambda a + \mu b$,

又 $n \perp a, n \perp b, \therefore n \cdot a = 0, n \cdot b = 0$

从而, $n \cdot c = n \cdot (\lambda a + \mu b)$

$$= \lambda(n \cdot a) + \mu(n \cdot b)$$

$$= 0. \quad \text{问题得证.}$$



例3 试证: 三角形的三高线交于一点.

设 $\triangle ABC$ 的 BC, AC 两边上的高交于 P 点,

要证 $\overrightarrow{PC} \perp \overrightarrow{AB}$, 为此

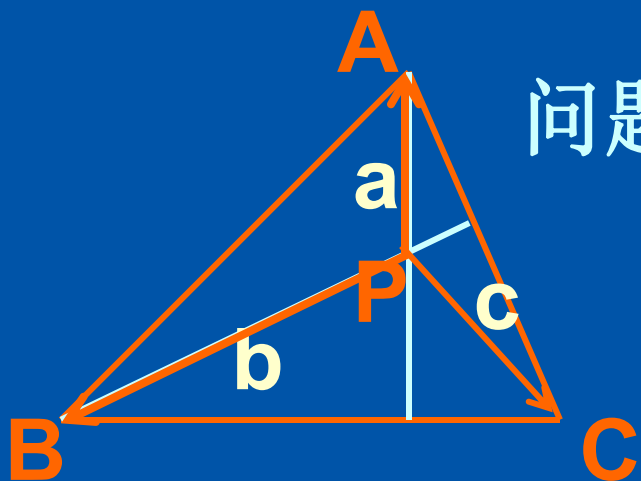
又设 $\overrightarrow{PA} = a, \overrightarrow{PB} = b, \overrightarrow{PC} = c,$

于是, $\overrightarrow{AB} = b - a, \overrightarrow{BC} = c - b, \overrightarrow{CA} = a - c;$

问题转化为证明: $c(b - a) = 0;$

$\because PA \perp BC, \therefore a(c - b) = 0,$

即 $ac - ab = 0,$ 从而 $ac = ab;$



例3 试证: 三角形的三高线交于一点.

设 $\triangle ABC$ 的 BC, AC 两边上的高交于 P 点,

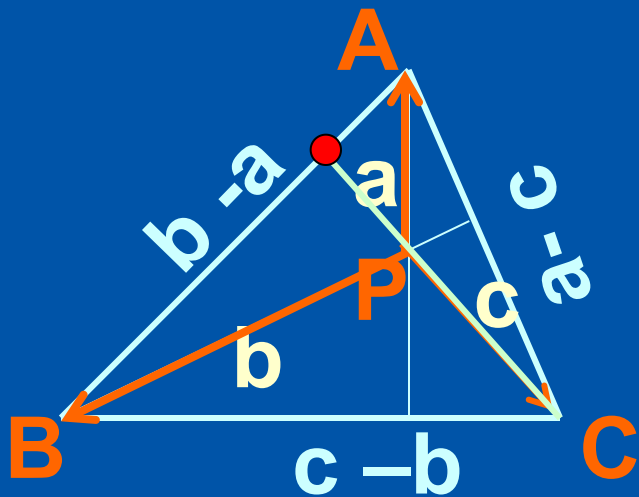
要证 $\vec{PC} \perp \vec{AB}$,

$\because \vec{PA} \perp \vec{BC}, \therefore a(c-b)=0$, 即 $ac - ab=0$, 从而 $ac = ab$

$\because \vec{PB} \perp \vec{AC}, \therefore b(c-a)=0$, 即 $bc = ab$;

于是, $bc = ac, c(b-a)=0$

即 $\vec{PC} \perp \vec{AB}$.

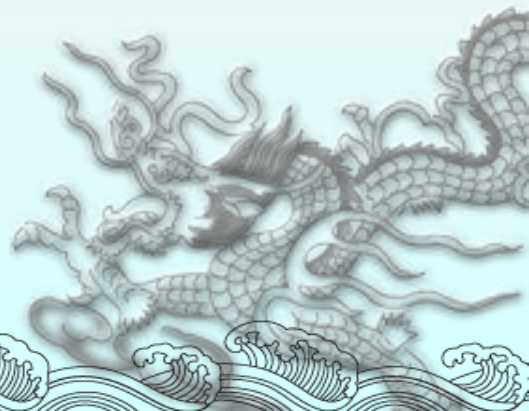


例 4 证明向量 \vec{c} 与向量 $(\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$ 垂直.

证

$$\begin{aligned} & [(\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}] \cdot \vec{c} \\ &= [(\vec{a} \cdot \vec{c})\vec{b} \cdot \vec{c} - (\vec{b} \cdot \vec{c})\vec{a} \cdot \vec{c}] \\ &= (\vec{c} \cdot \vec{b})[\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{c}] \\ &= 0 \end{aligned}$$

$$\therefore [(\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}] \perp \vec{c}$$



在直角坐标系 $\{0; i, j, k\}$ 下

用矢量分量进行数性积的有关运算



定理1.7.3 设 $a=X_1i+Y_1j+Z_1k$, $b=X_2i+Y_2j+Z_2k$, 则

$$a \cdot b = X_1X_2 + Y_1Y_2 + Z_1Z_2 \quad (1.7-6)$$

证: $a \cdot b = (X_1i + Y_1j + Z_1k)(X_2i + Y_2j + Z_2k)$

$$= X_1X_2(i \cdot i) + X_1Y_2(i \cdot j) + X_1Z_2(i \cdot k) +$$

$$+ Y_1X_2(j \cdot i) + Y_1Y_2(j \cdot j) + Y_1Z_2(j \cdot k) +$$

$$+ Z_1X_2(k \cdot i) + Z_1Y_2(k \cdot j) + Z_1Z_2(k \cdot k)$$

$$\because i \cdot i = 1, j \cdot j = 1, k \cdot k = 1, i \cdot j = 0, j \cdot k = 0, i \cdot k = 0$$

$$\therefore a \cdot b = X_1X_2 + Y_1Y_2 + Z_1Z_2 \quad \text{定理得证.}$$

两向量的数量积的应用

1. 数量积的坐标表达式

$$\text{设 } \vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}, \quad \vec{b} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}$$

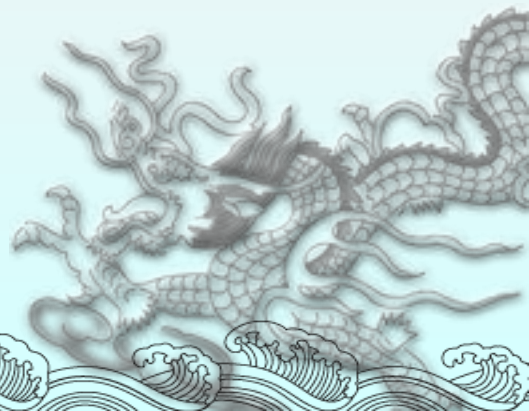
$$\vec{a} \cdot \vec{b} = (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \cdot (b_x \vec{i} + b_y \vec{j} + b_z \vec{k})$$

$$\because \vec{i} \perp \vec{j} \perp \vec{k}, \quad \therefore \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0,$$

$$\because |\vec{i}| = |\vec{j}| = |\vec{k}| = 1,$$

$$\therefore \vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1.$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$



由此，我们得到了如下结论：

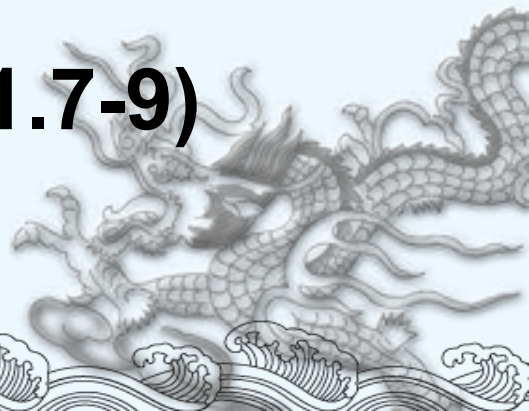
设 $\mathbf{a} = X_1\mathbf{i} + Y_1\mathbf{j} + Z_1\mathbf{k}$, $\mathbf{b} = X_2\mathbf{i} + Y_2\mathbf{j} + Z_2\mathbf{k}$ ，则

$$\mathbf{a} \cdot \mathbf{b} = X_1X_2 + Y_1Y_2 + Z_1Z_2 \quad (1.7-8)$$

并且还有

推论 设 $\mathbf{a} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ ，则

$$\mathbf{a} \cdot \mathbf{i} = X, \quad \mathbf{a} \cdot \mathbf{j} = Y, \quad \mathbf{a} \cdot \mathbf{k} = Z \quad (1.7-9)$$



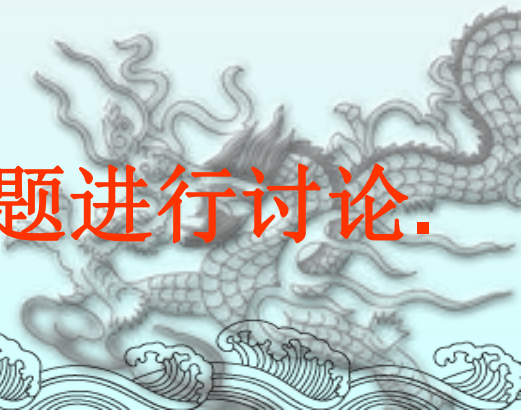
$$\vec{a}=X_1\mathbf{i}+Y_1\mathbf{j}+Z_1\mathbf{k}, \quad \vec{b}=X_2\mathbf{i}+Y_2\mathbf{j}+Z_2\mathbf{k}$$

$$\vec{a}\cdot\vec{b}=X_1X_2+Y_1Y_2+Z_1Z_2 \quad (1.7-6)$$

由公式(1.7-6)可以解决以下问题:

- 1) 两点间的距离
- 2) 矢量的方向余弦
- 3) 两矢量的交角

下面在直角坐标系下,分别对上述问题进行讨论.



1) 两点间的距离

$$\text{因为 } \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \angle(\mathbf{a}, \mathbf{b}) \quad (1.7-1)$$

$$\text{所以 } \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2, |\mathbf{a}| = \sqrt{a^2} .$$

定理1.7.4 设 $\mathbf{a} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, 则

$$|\mathbf{a}| = \sqrt{X^2 + Y^2 + Z^2} \quad (1.7-10)$$

证: 因为 $\mathbf{a} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, 且 $\mathbf{a} \cdot \mathbf{a} = X_1X_2 + Y_1Y_2 + Z_1Z_2$

$$\text{所以 } a^2 = X^2 + Y^2 + Z^2, \text{ 即 } |\mathbf{a}|^2 = X^2 + Y^2 + Z^2$$

定理得证.



定理1.7.5 空间两点 $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ 间的距离是

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad (1.7-11)$$

证: 因为 $\overrightarrow{P_1P_2} = \{ x_2 - x_1, y_2 - y_1, z_2 - z_1 \}$

所以 $d = |\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

定理得证.

空间两点间距离公式

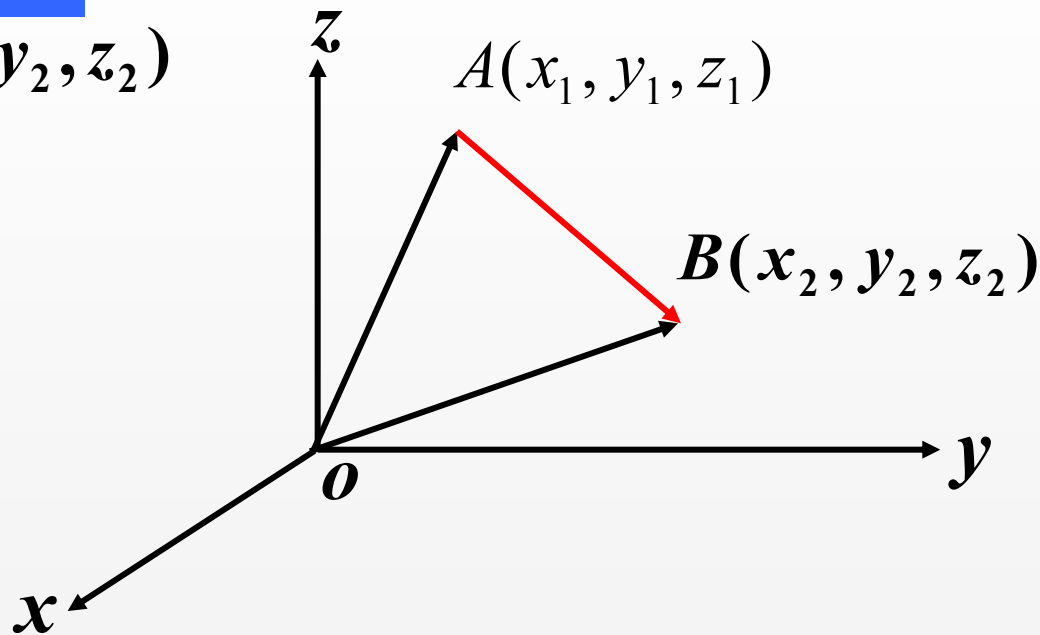
设 $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

为空间两点.

$$d = |\overrightarrow{AB}|$$

由 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \{x_2, y_2, z_2\} - \{x_1, y_1, z_1\} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\},$$



$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2. 向量模的坐标表达式

$$\begin{aligned}\vec{r} &= \overrightarrow{OM} \\ &= \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}\end{aligned}$$

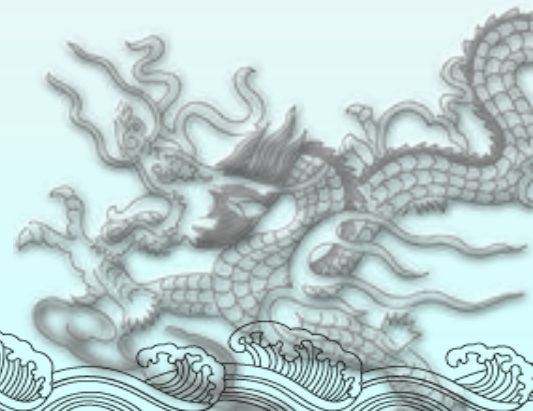
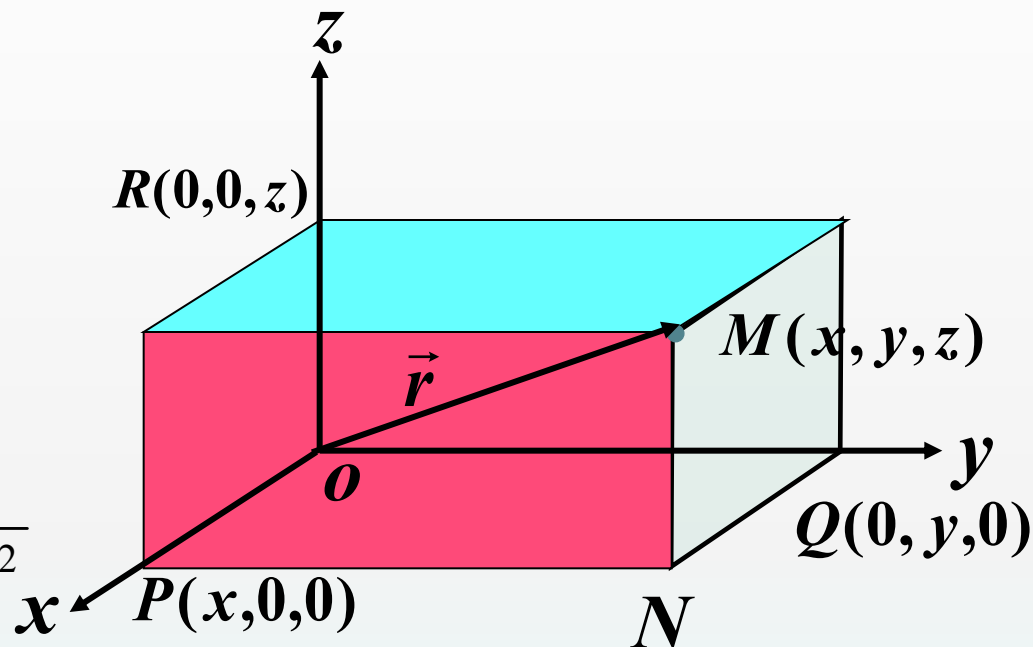
由勾股定理

$$\begin{aligned}|\vec{r}| &= |\overrightarrow{OM}| \\ &= \sqrt{|\overrightarrow{OP}|^2 + |\overrightarrow{OQ}|^2 + |\overrightarrow{OR}|^2}\end{aligned}$$

由 $\overrightarrow{OP} = x\vec{i}$, $\overrightarrow{OQ} = y\vec{j}$, $\overrightarrow{OR} = z\vec{k}$.

有 $|\overrightarrow{OP}| = |x|$, $|\overrightarrow{OQ}| = |y|$, $|\overrightarrow{OR}| = |z|$,

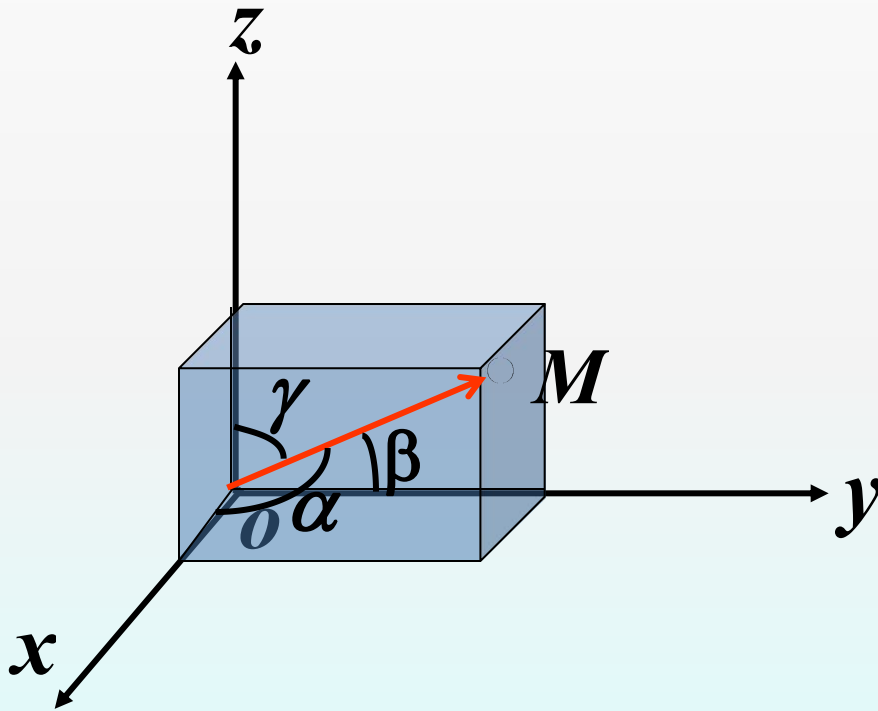
$$\therefore |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$



矢量的方向角

非零向量 \overrightarrow{OM} 的方向角: α 、 β 、 γ

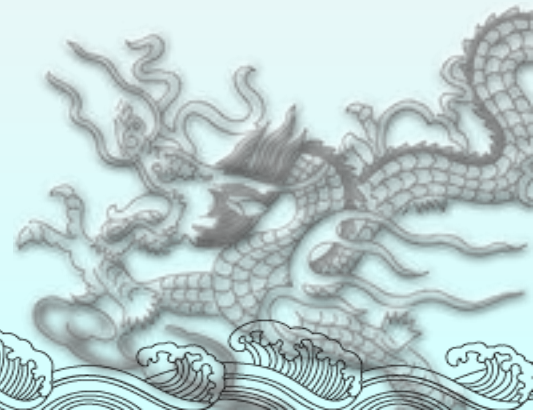
非零向量与三条坐标轴的正向的夹角称为**方向角**.



$$0 \leq \alpha \leq \pi,$$

$$0 \leq \beta \leq \pi,$$

$$0 \leq \gamma \leq \pi.$$



2) 矢量的方向余弦

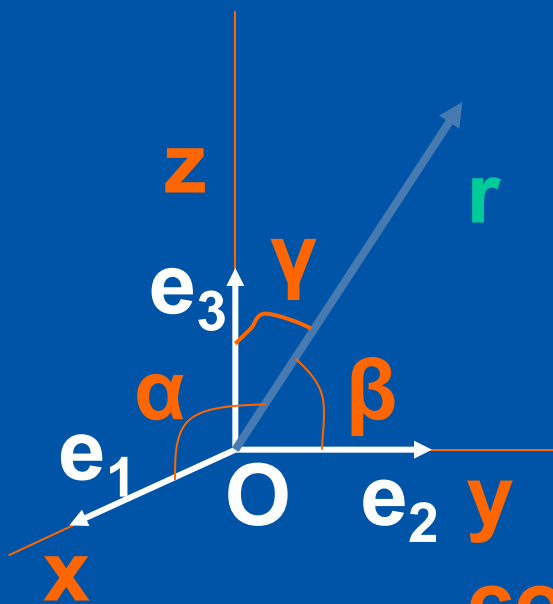
方向角的余弦叫做矢量的方向余弦.

矢量的方向完全由它的方向角确定.

$$\alpha = \angle(r, e_1), \beta = \angle(r, e_2),$$

$$\gamma = \angle(r, e_3);$$

α, β, γ 就是矢量 r 的三个方向角.



$\cos\alpha, \cos\beta, \cos\gamma$ 就是 r 的方向余弦.

定理1.7.6 非零向量 $\vec{a} = Xi+Yj+Zk$ 的方向余弦是

$$\cos\alpha = \frac{X}{|\vec{a}|} = \frac{X}{\sqrt{X^2+Y^2+Z^2}}$$

$$\cos\beta = \frac{Y}{|\vec{a}|} = \frac{Y}{\sqrt{X^2+Y^2+Z^2}}$$

$$\cos\gamma = \frac{Z}{|\vec{a}|} = \frac{Z}{\sqrt{X^2+Y^2+Z^2}}$$

且 $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\alpha = \angle(\vec{a}, \vec{e}_1), \beta = \angle(\vec{a}, \vec{e}_2), \gamma = \angle(\vec{a}, \vec{e}_3)$

证明： 因为 $\vec{a} \cdot \vec{i} = |\vec{a}| \cos\alpha$

又 $\vec{a} \cdot \vec{i} = X$ 所以 $X = |\vec{a}| \cos\alpha$

$$\text{从而 } \cos\alpha = \frac{X}{|\vec{a}|} = \frac{X}{\sqrt{X^2+Y^2+Z^2}}$$

$$\text{同理 } \cos\beta = \frac{Y}{|\vec{a}|} = \frac{Y}{\sqrt{X^2+Y^2+Z^2}}$$

$$\cos\gamma = \frac{Z}{|\vec{a}|} = \frac{Z}{\sqrt{X^2+Y^2+Z^2}}$$

且 $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

定理得证.

由定理证明过程可知:

$$X = |\vec{a}| \cos \alpha$$

$$Y = |\vec{a}| \cos \beta$$

$$Z = |\vec{a}| \cos \gamma$$

即任意一个矢量都由其模和方向余弦唯一确定.

且由 $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ 知

$$\vec{a}^0 = \left\{ \cos \alpha, \cos \beta, \cos \gamma \right\}$$

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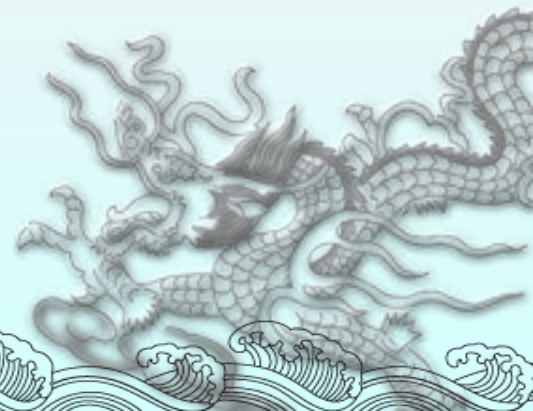
3) 两矢量的交角

定理1.7.7 如果空间中的两个非零矢量为

$$\vec{a} = \{X_1, Y_1, Z_1\} \quad \vec{b} = \{X_2, Y_2, Z_2\} \quad \text{则}$$

$$\cos \angle (\vec{a}, \vec{b}) = \frac{X_1 X_2 + Y_1 Y_2 + Z_1 Z_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2} \cdot \sqrt{X_2^2 + Y_2^2 + Z_2^2}}$$

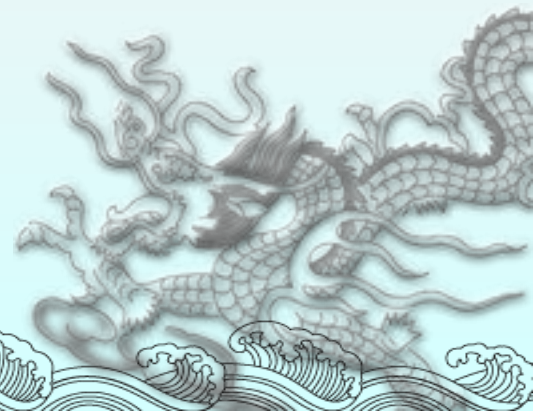
证明 思路 分析



$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \angle(\vec{a}, \vec{b})$$

从而

$$\cos \angle(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

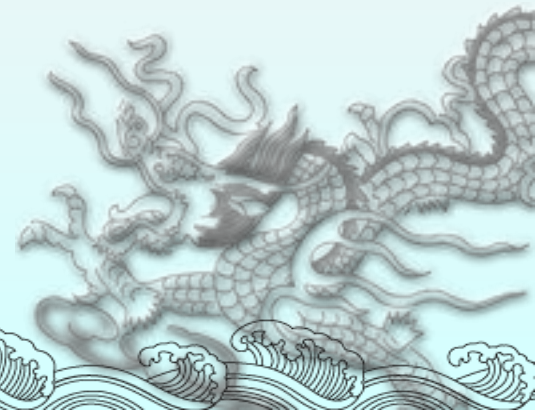


$$\vec{\mathbf{a}} = \{X_1, Y_1, Z_1\} \quad \vec{\mathbf{b}} = \{X_2, Y_2, Z_2\}$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = X_1 X_2 + Y_1 Y_2 + Z_1 Z_2$$

$$|\vec{\mathbf{a}}| = \sqrt{X_1^2 + Y_1^2 + Z_1^2}$$

$$|\vec{\mathbf{b}}| = \sqrt{X_2^2 + Y_2^2 + Z_2^2}$$



证: $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \angle(\vec{a}, \vec{b})$

又 \vec{a}, \vec{b} 为非零矢量,

所以 $\cos \angle(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$

另一方面 $\vec{a} \cdot \vec{b} = X_1 X_2 + Y_1 Y_2 + Z_1 Z_2$

$$|\vec{a}| = \sqrt{X_1^2 + Y_1^2 + Z_1^2} \quad |\vec{b}| = \sqrt{X_2^2 + Y_2^2 + Z_2^2}$$

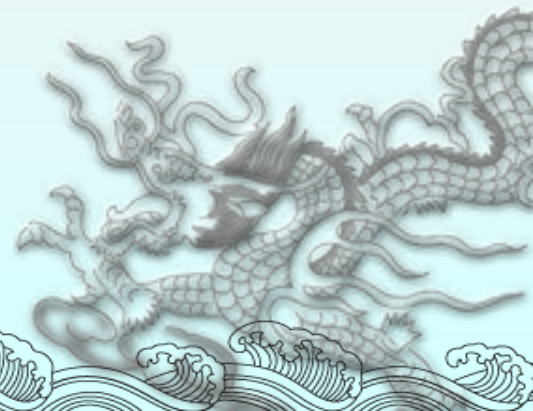
于是

$$\cos \angle(\vec{a}, \vec{b}) = \frac{X_1 X_2 + Y_1 Y_2 + Z_1 Z_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2} \cdot \sqrt{X_2^2 + Y_2^2 + Z_2^2}}$$

推论 如果 $\vec{a} = \{X_1, Y_1, Z_1\}$ $\vec{b} = \{X_2, Y_2, Z_2\}$

则 $\vec{a} \perp \vec{b}$ 的充要条件是

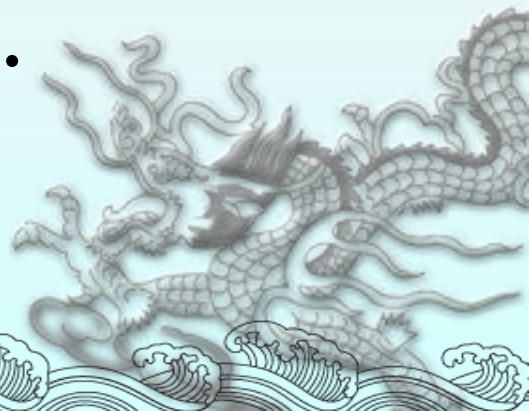
$$X_1X_2 + Y_1Y_2 + Z_1Z_2 = 0$$



例 5 已知 $\vec{a} = \{1, 1, -4\}$, $\vec{b} = \{1, -2, 2\}$, 求
(1) $\vec{a} \cdot \vec{b}$; (2) \vec{a} 与 \vec{b} 的夹角.

解 (1) $\vec{a} \cdot \vec{b} = 1 \cdot 1 + 1 \cdot (-2) + (-4) \cdot 2 = -9$.

$$(2) \cos \theta = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \sqrt{b_x^2 + b_y^2 + b_z^2}}$$
$$= -\frac{1}{\sqrt{2}}, \quad \therefore \theta = \frac{3\pi}{4}.$$



在平面直角坐标系下也有类似结论.

设 $\vec{a} = \{X_1, Y_1\}$ $\vec{b} = \{X_2, Y_2\}$ 则

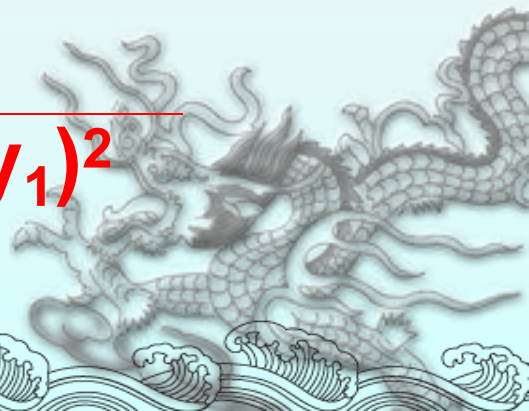
$$\vec{a} \cdot \vec{b} = X_1 X_2 + Y_1 Y_2$$

$$\vec{a} \cdot \vec{i} = X_1 \quad \vec{a} \cdot \vec{j} = Y_1$$

$$|\vec{a}| = \sqrt{X_1^2 + Y_1^2}$$

两点 $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ 间的距离

$$d = |\vec{P_1 P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

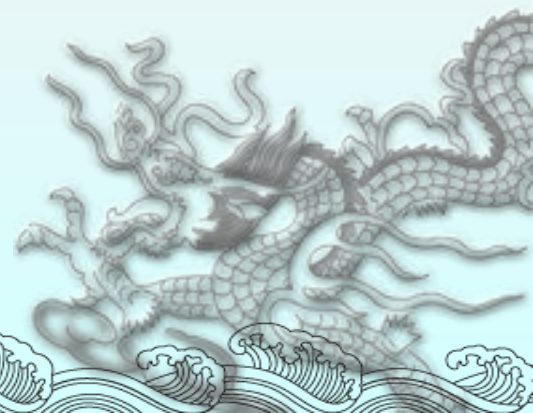


设 $\cos\alpha, \cos\beta$ 是 \vec{a} 的方向余弦. 则

$$\cos\alpha = \frac{X_1}{|\vec{a}|} = \frac{X_1}{\sqrt{X_1^2 + Y_1^2}}$$

$$\cos\beta = \frac{Y_1}{|\vec{a}|} = \frac{Y_1}{\sqrt{X_1^2 + Y_1^2}}$$

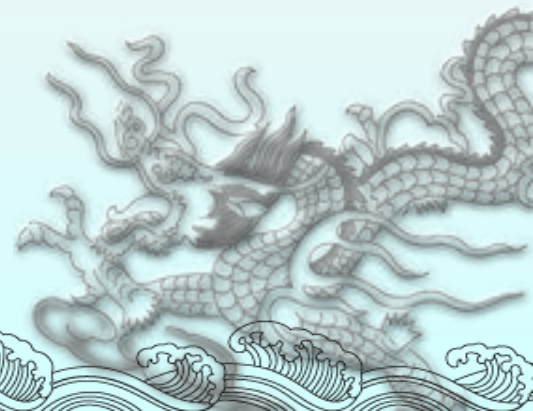
且 $\cos^2\alpha + \cos^2\beta = 1$



$$\cos \angle (\vec{a}, \vec{b}) = \frac{X_1 X_2 + Y_1 Y_2}{\sqrt{X_1^2 + Y_1^2} \cdot \sqrt{X_2^2 + Y_2^2}}$$

$\vec{a} \perp \vec{b}$ 的充要条件是

$$X_1 X_2 + Y_1 Y_2 = 0$$



例6 已知三点A(1,0,0),B(3,1,1),C(2,0,1)且

$$\overrightarrow{BC} = \vec{a}, \quad \overrightarrow{CA} = \vec{b}, \quad \overrightarrow{AB} = \vec{c},$$

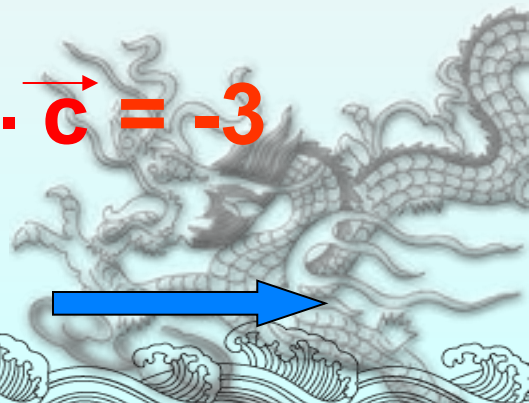
求 1) $\angle(\vec{a}, \vec{b})$; 2) 射影 _{\vec{c}} \vec{a} .

解 $\vec{a} = \{-1, -1, 0\}$, $\vec{b} = \{-1, 0, -1\}$,

$$\vec{c} = \{2, 1, 1\}, \quad |\vec{a}| = \sqrt{2}, \quad |\vec{b}| = \sqrt{2}$$

$$|\vec{c}| = \sqrt{6}$$

$$\vec{a} \cdot \vec{b} = (-1)(-1) + (-1)0 + 0 \cdot (-1) = 1, \quad \vec{a} \cdot \vec{c} = -3$$



$$\cos \angle (\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

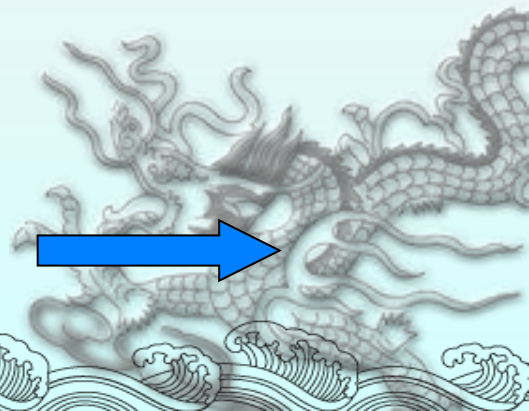
$$\vec{a} \cdot \vec{c} = |\vec{c}| \text{射影}_{\vec{c}} \vec{a} \quad \text{射影}_{\vec{c}} \vec{a} = \frac{\vec{a} \cdot \vec{c}}{|\vec{c}|}$$

因此,要求出

$$\vec{a} \cdot \vec{b} \quad \vec{a} \cdot \vec{c} \quad |\vec{a}| \quad |\vec{b}| \quad |\vec{c}|$$

$$\vec{a} \cdot \vec{b} = X_1 X_2 + Y_1 Y_2 + Z_1 Z_2$$

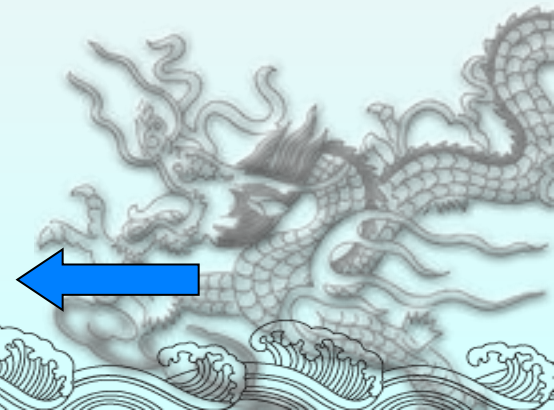
$$|\vec{a}| = \sqrt{X_1^2 + Y_1^2 + Z_1^2}$$



A(1,0,0), B(3,1,1), C(2,0,1)

$$\overrightarrow{BC} = \vec{a} = \{X_C - X_B, Y_C - Y_B, Z_C - Z_B\},$$

$$|\vec{a}| = \sqrt{X_1^2 + Y_1^2 + Z_1^2}$$



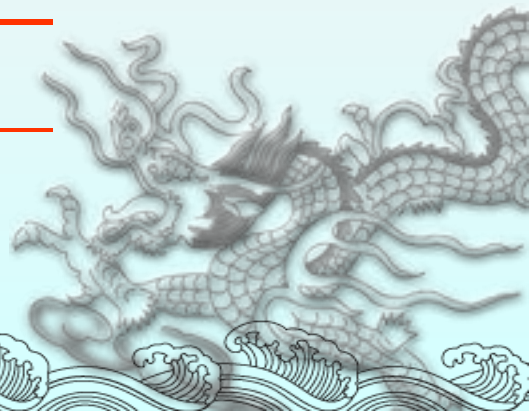
即 $|\vec{a}| = \sqrt{2}$ $|\vec{b}| = \sqrt{2}$ $|\vec{c}| = \sqrt{6}$

$\vec{a} \cdot \vec{b} = 1$, $\vec{a} \cdot \vec{c} = -3$ 所以

$$\cos \angle (\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{1}{\sqrt{2} \sqrt{2}} = \frac{1}{2}$$

$$\therefore \angle (\vec{a}, \vec{b}) = \frac{\pi}{3}$$

$$\text{射影}_{\vec{c}} \vec{a} = \frac{\vec{a} \cdot \vec{c}}{|\vec{c}|} = \frac{-3}{\sqrt{6}} = -\frac{\sqrt{6}}{2}$$



例7 利用数性积证明柯西. 施瓦兹不等式:

$$(a_1b_1+a_2b_2+a_3b_3)^2 \leq (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

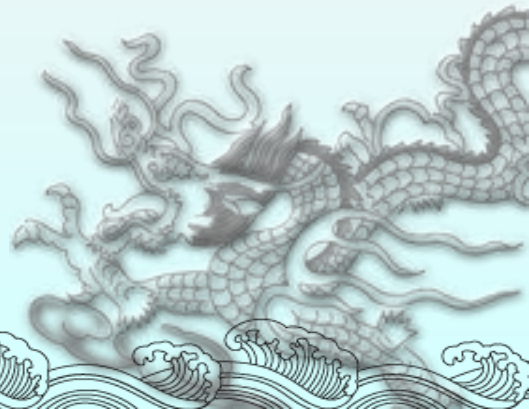
如果令 $\vec{a} = \{a_1, a_2, a_3\}$, $\vec{b} = \{b_1, b_2, b_3\}$, 则

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2, \quad |\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2$$

问题转化为证明
不等式成立:

$$(\vec{a} \cdot \vec{b})^2 \leq |\vec{a}|^2 \cdot |\vec{b}|^2$$



例7 利用数性积证明柯西. 施瓦兹不等式:

$$(a_1b_1+a_2b_2+a_3b_3)^2 \leq (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

证: 令 $\vec{a} = \{a_1, a_2, a_3\}$, $\vec{b} = \{b_1, b_2, b_3\}$,

因为 $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \angle(\vec{a}, \vec{b})$

$$-1 \leq \cos \angle(\vec{a}, \vec{b}) \leq 1$$

所以 $(\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 \cos^2 \angle(\vec{a}, \vec{b})$

于是 $(\vec{a} \cdot \vec{b})^2 \leq |\vec{a}|^2 |\vec{b}|^2$ 问题得证.

课堂小结

1. 两向量的数量积是一个数量；
2. 数量积在向量的模、两点间的距离、两向量的夹角等方面的应用。



思考题：

两向量相乘什么情况下结果

是向量？

